Glauber dynamics in spin models with non-symmetric interactions

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PLAN

1. Spin models
2. Symmetric and non-symmetric coupling
3.Glauber dynamics
4. Phase transitions
5. Number of local attractors may be exponentially large
6. Chaos 7. Applications

Spin models

The states of the systems under consideration are spin patterns $\mathbf{s} = (s_1, s_2, ..., s_N) \in S^N$, where $s_i \in S = \{-1, 1\}$. The dynamics was introduced by R. J. Glauber. The spin states define a vector θ (a field acting on spins) with components θ_i by

$$\theta_i = \sum_{j=1}^{N} J_{ij} s_j + h_i, \tag{1}$$

where the real-valued coefficients J_{ij} define a coupling, in general, asymmetric, and h_i are constants describing external fields.

Spin models

The parallel Glauber dynamics is a Markov chain with transition probabilities defined by

$$\Pr[\mathbf{s}(t+1)|\mathbf{s}(t)] = \frac{\exp\left(\beta \sum_{i=1}^{N} s_i(t+1)\theta_i(t)\right)}{\prod_{i=1}^{N} 2\cosh(\beta\theta_i(t))}, \qquad (2)$$

where β is a parameter (the inverse temperature). In this spin dynamics, at each step, all spins can be flipped with a certain probability as defined by Eq. (2).

Applications

The Ising and spin glass models have received much attention in statistical mechanics. Different methods, such as the mean-field theory, the replica symmetric ansatz, and the cavity method, have been developed to understand phase transitions, frustrated phenomena, and other intriguing effects (see book of M. Talagrand). These models have formidable applications in neural network theory (J.J. Hopfield, E. Gardner), computation theory (Moore-Mertens), and machine learning problems.
Glauber dynamics, originating as a stochastic model for the evolution of spin systems, has found applications far beyond its initial context. It plays a crucial role in understanding discrete stochastic processes, particularly in network science and mathematics. Its utility extends to Markov Chain Monte Carlo (MCMC) methods, graph theory, and combinatorial optimization.

Ising model

The famous Ising model (1920) considers interactions only between nearest spins, the corresponding Hamiltonian has the form

$$\mathbf{H} = \sum_{i,j \in B} J_{ij} s_i s_i + \sum_{j \in B} h_j s_j \tag{3}$$

where i, j lie on a bounded subset B of the d-dimensional lattice $\mathbb{Z}^d$ and symmetric coefficients $J_{i,j} \neq 0$ if i, j are nearest neighbors. A number of celebrated works are devoted to this model of statistical physics (H. Kramers, L. Onsager, and many others). The main effect is that for $d > 1$ and for small inverse temperatures $\beta > 0$ we have a disordered state, and for large β - an ordered one.

Spin glass and neural networks

In contrast to Ising model, in spin glass models coupling J_{ij} involves all nodes i, j (not only nearest ones). They were studied beginning with 1980-s. The famous Hopfield model (1982) uses the following representation:

$$J_{ij} = \sum_{k=1}^{p} a_{ik} a_{jk} \tag{4}$$

where $p << N$. So, we have a globally connected spin network with symmetric coupling. It allows us to explain the mechanism of associative memory by the dynamical system theory. Further this model was used J. Reinitz et al. to explain the morphogenesis. It can be applied for cancer epigenetics, for ecology, etc.

Layered patterns

Figure: Zebra pattern and other layered patterns can be explained by Turing and Reinitz et al. models

PROBLEM!!

J. J. Hopfield used symmetric globally connected model, but real interactions between neurons (genes) are local and asymmetric. What can happened then ?? For example, that one can say about the Glauber dynamics in such systems? It was an open problem. For symmetric case there was many results.

SOLUTION !!

We use a generalization of J. J. Hopfield formula:

$$J_{ij} = \sum_{k}^{p} a_{ik} b_{kj} \tag{5}$$

where $p << N$. Then in limit $N \to \infty$ we obtain the dynamical system defined by iterations.

Dynamical system

$$\bar{q}_l(t+1) = \sum_{j=1}^{N} b_{lj} \tanh\Big(\beta\theta_j(\bar{q}(t))\Big), \tag{6}$$

$$\theta_j(\bar{q}) = \sum_{l=1}^{p} a_{jl}\bar{q}_l + h_j. \tag{7}$$

Dynamical system

In these relations,

$$q_l = \sum_{j=1}^{N} b_{lj} s_j, \tag{8}$$

$$\bar{q}_l = E[q_l]. \tag{9}$$

One can show that the probabilities of deviations of q_l from the averages $\bar{q}_l$ are exponentially small as $N \to \infty$, where N is the spin number.

Using known results

Furthermore we use known results for system (8), (9). This system enjoys the property of Universal Dynamical Approximation, which generalizes the property of Universal Approximation for neural networks. Roughly speaking, these systems can simulate any dynamical systems by adjusting their parameters.

Main Theorem

Theorem I. (S Vakulenko, D. Grigoriev and Sudakov, 2025) *Syst. (8), (9) can generate all possible finite dimensional structurally stable dynamics when we vary their parameters (N, a, b and h). In particular, these systems are capable to generate hyperbolic chaotic dynamics, such as Smale horseshoe, Anosov flow, etc. These chaotic sets can be at any dimension*.

Simple way to chaos

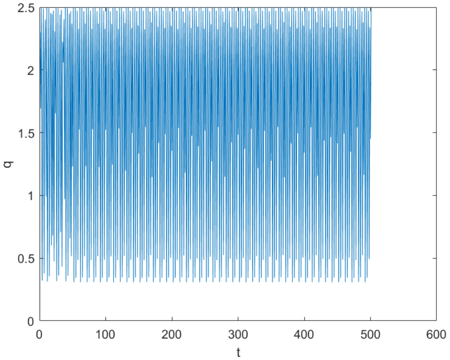
We describe how to apply the celebrated Sharkovskii theorem to describe changes in the dynamics of the system (8), (9) as $\beta > 0$ increases.

We recall the Sharkovskii pioneering result (1960). Consider maps $v \to f(v)$ of the interval $[0, 1]$ into itself defined by a continuous function f. The Sharkovskii theorem, in particular, asserts that the existence of a point v_* of period 3 implies existence of points of all odd periods $2m + 1$ (the point v_* has the period n if $f^{(n)}(v_*) = v_*$, where $f^{(n)}$ denotes the n-th iteration of f). Since the existence of infinitely many periodical trajectories is a feature of chaos (it holds for hyperbolic sets), it is sometimes said briefly that the existence of a point of period three causes chaos.

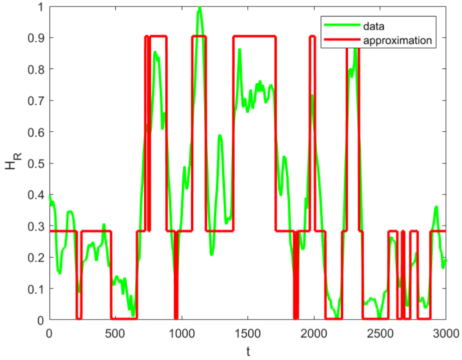
Simple way to chaos

Proposition For all $N > 1$ there exist coefficients b_j, h_j, a_j such that for sufficiently small positive β all the trajectories of the map $\bar{q} \to f_{\beta,a,b,h}(\bar{q})$ are convergent to a rest point and for sufficiently large $\beta > \beta_0 > 0$ this map has a point q_* of the period 3.

Numerical simulation



Sharkovski chaos in Glauber dynamics, obtained numerically.



Approximation of the heart rate by a Glauber dynamics.

Conclusions

1. Spin systems with non-symmetric interaction are capable generate complex dynamics;
2. They exhibit complex bifurcations as the inverse temperature changes;
3. They can have applications to cluster analysis, cancer epigenetics modeling etc.

Publications

[1] Sergey Vakulenko, Dmitry Grigoriev, and Ivan Sudakov Non-equilibrium dynamics in complex networks via asymmetric Glauber models. J. Phys. A Math and General, 2025.