Ramanujan Type Formula for Polylogarithm

Stepan Samulevich

Abstract. The paper proposes a generalization of the well-known Ramanujan formula for the Riemann zeta function to the case of the polylogarithm. The resulting formulas allow the acceleration of the convergence of the original series to exponential speed. As a consequence, formulas for the polygamma functions and many other special functions can be derived

Introduction

The polylogarithm is a special function defined by the Dirichlet series

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

In this work, we consider the values on the unit circle $\mathrm{Li}_k(e^{2\pi i\lambda})$. A particular case of the polylogarithm is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for which a general formula for even values is known in terms of Bernoulli numbers:

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}}{2(2k)!}$$

For odd values of the Riemann zeta function, Ramanujan's formula [1] provides good approximations:

$$\zeta(4k-1) = (2\pi)^{4k-1} \sum_{m=0}^{2k} \frac{(-1)^{m+1}B_{2m}B_{4k-2m}}{2(2m)!(4k-2m)!} - 2\sum_{n=1}^{\infty} \frac{1}{n^{4k-1}(e^{2\pi n}-1)}$$

There also exists a similar formula for $\zeta(4k+1)$

A large number of generalizations of this formula are known [2]. In the present work, the values of the polylogarithm on the unit circle $\mathrm{Li}_k(e^{2\pi i\lambda})$ are expressed in terms of Bernoulli polynomials. However, the convergence rate of the remainder series in the resulting formulas depends on λ: the asymptotics of the general term is $O(e^{2\pi i\sqrt{\lambda}n})$ for $\lambda \in (0, \frac{1}{2}]$ and $O(e^{2\pi i\sqrt{1-\lambda}n})$ for $\lambda \in (\frac{1}{2}, 1)$ The closer the point is to -1, the faster the convergence; the farther it is, the slower.

To overcome this difficulty, the well-known duplication formula for the polylogarithm was applied:

$$\operatorname{Li}_s(z) + \operatorname{Li}_s(-z) = 2^{1-s} \operatorname{Li}_s(z^2)$$

For small arguments, this formula allows computing the value of the polylogarithm at points with better asymptotics: $-z$ and z^2. Applying it sufficiently many times yields an asymptotic of the general term of the remainder series as $O(e^{-\pi n})$ for any λ.

1. Main results

Theorem 1. $\forall\lambda \in (0, \frac{1}{2}],\ k \in \mathbb{N}$

$$\operatorname{Li}_{2k}(e^{2\pi i\lambda}) = \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}(\lambda)}{2(2k)!} + \frac{(2\pi\sqrt{\lambda})^{2k}}{\sqrt{\lambda}}i\sum_{m=0}^{k}\frac{(-1)^{m+1}B_{2m+1}(\lambda)B_{2(k-m)}}{\lambda^m(2m+1)!(2(k-m))!} - \\ -i\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\left(\frac{(-1)^k\lambda^k\sinh\frac{(2\lambda-1)\pi n}{\sqrt{\lambda}}}{\sqrt{\lambda}\sinh\frac{\pi n}{\sqrt{\lambda}}} + \frac{2\sin(2\pi\lambda n)}{e^{2\pi n\sqrt{\lambda}}-1}\right)$$

Theorem 2. $\forall\lambda \in (0, \frac{1}{2}],\ k \in \mathbb{N}$

$$\operatorname{Li}_{2k+1}(e^{2\pi i\lambda}) = (2\pi\sqrt{\lambda})^{2k+1}\sum_{m=0}^{k}\frac{(-1)^{m+1}B_{2m}(\lambda)B_{2(k-m)}}{\lambda^m(2m)!(2(k-m)!)} + \frac{(-1)^{k+1}(2\pi)^{2k+1}B_{2k+1}(\lambda)}{2(2k+1)!}i - \\ -\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\left(\frac{(-1)^k\lambda^k\cosh\frac{(2\lambda-1)\pi n}{\sqrt{\lambda}}}{\sinh\frac{\pi n}{\sqrt{\lambda}}} + \frac{2\cos(2\pi\lambda n)}{e^{2\pi n\sqrt{\lambda}}-1}\right)$$

Remark 1. *It is sufficient to calculate* $\operatorname{Li}_k(e^{2\pi i\lambda})$ *only for* $\lambda \in (0, \frac{1}{2}]$ *due to the identity* $\operatorname{Li}_k(e^{-2\pi i\lambda}) = \overline{\operatorname{Li}_k(e^{2\pi i\lambda})}$

Theorem 3. $\forall\lambda \in (0, \frac{1}{4}],\ s \in \mathbb{C}$

$$\operatorname{Li}_s(e^{2\pi i\lambda}) = 2^{1-s}\left(2^{\lceil -2-\log_2\lambda\rceil}\operatorname{Li}_s\left(e^{2^{\lceil -2-\log_2\lambda\rceil+1}\pi i\lambda}\right) - \sum_{k=1}^{\lceil -2-\log_2\lambda\rceil}2^{k-1}\operatorname{Li}_s\left(e^{\pi i(2^k\lambda+1)}\right)\right)$$

Remark 2. *Here every argument lies between* $\frac{\pi}{2}$ *and* $\frac{3\pi}{2}$*, so the asymptotic of the general term of the remainder series is not more than* $O(e^{-\pi n})$*.*

2. Approximation of sums of rational functions

The obtained formulas allow expressing rational values of the polygamma function

$$\psi^{(m)}(z) = (-1)^{m+1}m!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{m+1}},\ m \geq 1$$

using a Fourier sum

$$\psi^{(m)}\left(\frac{p}{q}\right) = \frac{(-1)^{m+1}q^m}{m!}\sum_{k=1}^{q}e^{-\frac{2\pi ipk}{q}}\operatorname{Li}_{m+1}\left(e^{\frac{2\pi ik}{q}}\right)$$

where $0 < p < q,\ m \geq 1$. For the digamma function, Gauss's formula [3] is known

$$\psi\left(\frac{p}{q}\right) = -\gamma - \ln 2q - \frac{\pi}{2}\operatorname{ctg}\frac{\pi p}{q} + 2\sum_{n=1}^{\lceil \frac{q}{2}\rceil - 1}\cos\frac{2\pi pn}{q}\ln\sin\frac{\pi n}{q}$$

where $0 < p < q$, γ is the Euler–Mascheroni constant. Thus, approximate formulas for sums of rational functions with rational poles can be derived due to the identity

$$\sum_{n=0}^{\infty}\sum_{k=1}^{m}\frac{a_k}{(n+b_k)^{r_k}} = \sum_{k=1}^{m}\frac{(-1)^{r_k}}{(r_k-1)!}a_k\psi^{(r_k-1)}(b_k)$$

Conclusion

In this work, we derived explicit expressions for the polylogarithm function on the unit circle in terms of Bernoulli polynomials and rapidly converging series. By applying the duplication formula iteratively, we obtained uniform asymptotic estimates for the remainder terms, improving the convergence rate to $O(e^{-\pi n})$ regardless of λ. This allows for efficient numerical evaluation of the polylogarithm at points on the unit circle.

Moreover, the obtained representations enabled the derivation of formulas for rational values of the polygamma function using discrete Fourier transform techniques. These formulas generalize classical results such as Gauss's digamma formula and provide new tools for analyzing and approximating sums of rational functions with rational poles.

References

[1] B.C. Berndt and A. Straub. Ramanujan's formula for $\zeta(2n{+}1)$. 2017, arXiv:1701.02964 [math.NT].

[2] B.C. Berndt, Ramanujan's Notebooks, Part II, Springer-Verlag, New York, 1989. pp. 276-296.

[3] Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. (1953). Higher Transcendental Functions, Vol. 1. McGraw-Hill.

Stepan Samulevich
SPbETU "LETI"
Saint Petersburg, Russia
e-mail: `stepansamulevic@gmail.com`