Ramanujan-type formulas for Cl-type Clausen functions and applications

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Definitions

Bernoulli numbers

Can be defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

Bernoulli polynomials

Can be defined by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Note: $B_n(0) = B_n$

Definitions

Riemann-zeta function

For $\Re(s) > 1$ $\zeta(s)$ is defined by the so called Dirichlet series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

Clausen functions

For $m \in \mathbb{N}$ standard Clausen functions are defined by the following Fourier series

- SL-type: $\mathrm{Sl}_{2m-1}(\theta) = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^{2m-1}}$, $\mathrm{Sl}_{2m}(\theta) = \sum_{k=1}^{\infty} \frac{\cos k\theta}{k^{2m}}$
- Cl-type: $\mathrm{Cl}_{2m-1}(\theta) = \sum_{k=1}^{\infty} \frac{\cos k\theta}{k^{2m-1}}$, $\mathrm{Cl}_{2m}(\theta) = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^{2m}}$

Classic results

Riemann-zeta function

Even values of the Riemann zeta-function are closely connected to the Bernoulli numbers

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2(2m)!}$$

Sl-type Clausen functions

Values of the Sl-type Clausen functions are closely connected to the Bernoulli polynomials

$$\mathrm{Sl}_{2m}(\theta) = \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}(\frac{\theta}{2\pi})}{2(2m)!}$$

$$\mathrm{Sl}_{2m-1}(\theta) = \frac{(-1)^{m}(2\pi)^{2m-1}B_{2m-1}(\frac{\theta}{2\pi})}{2(2m-1)!}$$

Examples

- $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
- $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$
- $\mathrm{Sl}_1(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{\pi}{2} - \frac{\theta}{2}$
- $\mathrm{Sl}_2(\theta) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \frac{\pi^2}{6} - \frac{\pi\theta}{2} + \frac{\theta^2}{4}$

Main result

Ramanujan formula (well-known)

Values of the Riemann-zeta function in $4k-1$ can be expressed via Bernoulli numbers

$$\zeta(4m-1)=(2\pi)^{4m-1}\sum_{k=0}^{2m}\frac{(-1)^{k+1}B_{2k}B_{4m-2k}}{2(2k)!(4m-2k)!}-2\sum_{n=1}^{\infty}\frac{1}{n^{4m-1}(e^{2\pi n}-1)}$$

Cl-type Clausen functions

Values of the Cl-type Clausen functions can be expressed via Bernoulli polynomials

$$\mathrm{Cl}_{2m}(\theta)=(2\pi)^{2m}\sum_{k=0}^{m}\frac{(-1)^{k+1}B_{2k+1}\left(\frac{\theta}{2\pi}\right)B_{2m-2k}}{(2k+1)!(2m-2k)!}-2\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^{2m}(e^{2\pi n}-1)}+\sum_{n=1}^{\infty}\frac{(-1)^m\sinh((\pi-\theta)n)}{n^{2m}\sinh\pi n}$$

$$\mathrm{Cl}_{2m-1}(\theta)=(2\pi)^{2m-1}\sum_{k=0}^{m}\frac{(-1)^{k+1}B_{2k}\left(\frac{\theta}{2\pi}\right)B_{2m-2k}}{(2k)!(2m-2k)!}-2\sum_{n=1}^{\infty}\frac{\cos n\theta}{n^{2m-1}(e^{2\pi n}-1)}-\sum_{n=1}^{\infty}\frac{(-1)^m\cosh((\pi-\theta)n)}{n^{2m-1}\sinh\pi n}$$

Examples

- $\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)}$
- $\zeta(7) = \frac{19\pi^7}{56700} - 2\sum_{n=1}^{\infty} \frac{1}{n^7(e^{2\pi n}-1)}$
- $\mathrm{Cl}_2(\theta) = \frac{\pi^2}{6} - \frac{\theta^2}{4} + \frac{\theta^3}{12\pi} - 2\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2(e^{2\pi n}-1)} - \sum_{n=1}^{\infty} \frac{\sinh((\pi-\theta)n)}{n^2 \sinh \pi n}$
- $\mathrm{Cl}_3(\theta) = \frac{7\pi^3}{90} - \frac{\pi^2\theta}{6} + \frac{\theta^3}{12} - \frac{\theta^4}{48\pi} - 2\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^3(e^{2\pi n}-1)} - \sum_{n=1}^{\infty} \frac{\cosh((\pi-\theta)n)}{n^3 \sinh \pi n}$

Convergence rate

Problem

Convergence rate in these formulas is dependent on θ. General term of the hyperbolic series can be expressed as $O(e^{\alpha(\theta)n})$. In the original formulas $\alpha(\theta) = -\theta$ for $\theta \in [0, \pi]$ and $\alpha(\theta) = 2\pi - \theta$ for $\theta \in [\pi, 2\pi]$.

Modification

The Lambert series can be adjusted to share part of its convergence rate, so that both series have $\alpha(\theta) = -\sqrt{2\pi\theta}$ for $\theta \in [0, \pi]$, and $\alpha(\theta) = -\sqrt{2\pi(2\pi - \theta)}$ for $\theta \in [\pi, 2\pi]$.

Convergence rate

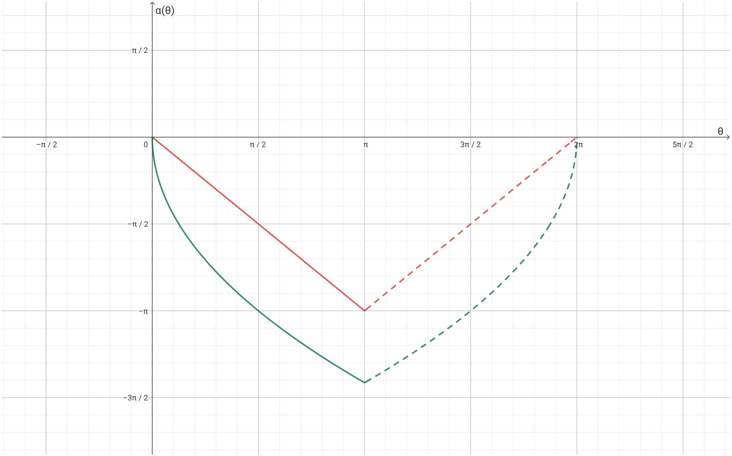

Figure: $\alpha(\theta)$ for $\theta \in [0, 2\pi]$ old – red, new – green

Polylogarithm

Definition

For $\Re(s) > 1,\ |z| \leq 1$ the polylogarithm of order s is defined by the Dirichlet series

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

Relation to the Clausen functions

On the unit circle polylogarithm can be expressed in terms of the Clausen functions

- $\mathrm{Li}_{2m+1}(e^{i\theta}) = \mathrm{Cl}_{2m+1}(\theta) + i\mathrm{Sl}_{2m+1}(\theta)$
- $\mathrm{Li}_{2m}(e^{i\theta}) = \mathrm{Sl}_{2m}(\theta) + i\mathrm{Cl}_{2m}(\theta)$

Convergence rate

Convergence rate

Convergence rate of the polylogarithm is the same as that for the Clausen functions due to their relation.

Duplication formula

For polylogarithm there exists the duplication formula

$$\mathrm{Li}_s(z) + \mathrm{Li}_s(-z) = 2^{1-s}\mathrm{Li}_s(z^2)$$

which on the unit circle can be re-written as

$$\mathrm{Li}_s(e^{i\theta}) + \mathrm{Li}_s(e^{i(\theta+\pi)}) = 2^{1-s}\mathrm{Li}_s(e^{2i\theta})$$

Applied sufficiently many times, this can lead to $\forall\theta\ \alpha(\theta) \leq -\pi$.

Duplication formula

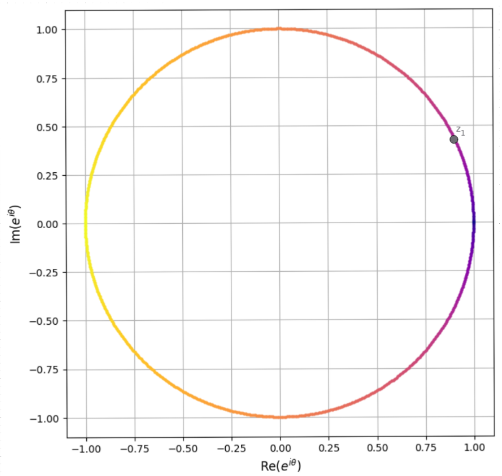

Figure: Heat map of the convergence rate of the polylogarithm

Duplication formula

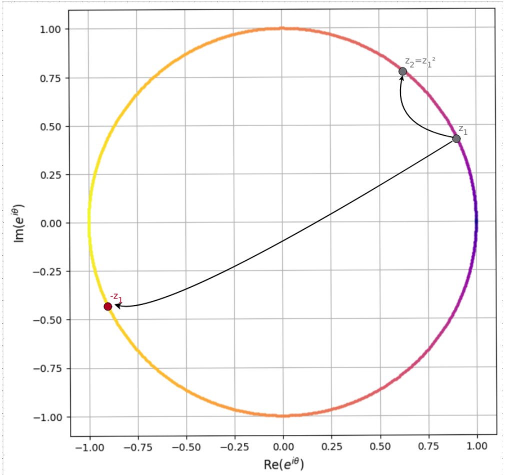

Figure: Heat map of the convergence rate of the polylogarithm

Duplication formula

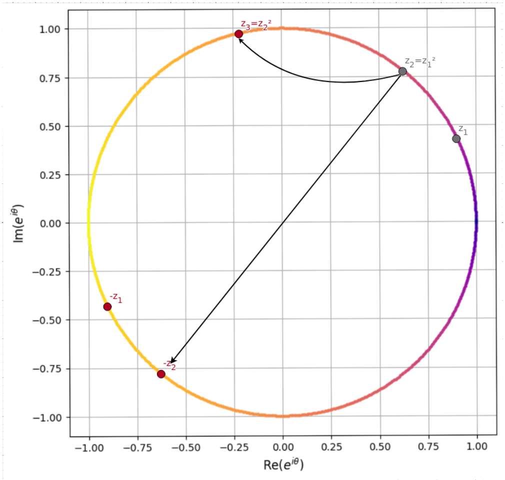

Figure: Heat map of the convergence rate of the polylogarithm

Convergence rate

Result

With this technique only values on the left half of the circle should be calculated, where $\alpha(\theta) \leq -\pi$.

Clausen functions

The same trick can be applied to calculate Clausen functions due to their own duplication formula

$$\mathrm{Cl}_m(\theta) - (-1)^m \mathrm{Cl}_m(\pi - \theta) = 2^{1-m} \mathrm{Cl}_m(2\theta)$$

Other applications

Polygamma functions

Polygamma functions are difined by the derivatives of the logarithm of the gamma function

$$\psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z)$$

Relation to the polylogarithm

Polylogarithm is related to the polygamma functions by the Fourier sum

$$\psi^{(m)}\left(\frac{p}{q}\right) = \frac{(-1)^{m+1} q^m}{m!} \sum_{k=1}^{q} e^{-\frac{2\pi i p k}{q}} \mathrm{Li}_{m+1}\left(e^{\frac{2\pi i k}{q}}\right)$$

where $0 < p < q$.

Other applications

Sums of rational functions

Sums of rational functions can be expressed in terms of the polygamma functions

$$\sum_{n=0}^{\infty}\sum_{k=1}^{m}\frac{a_k}{(n+b_k)^{r_k}}=\sum_{k=1}^{m}\frac{(-1)^{r_k}}{(r_k-1)!}a_k\psi^{(r_k-1)}(b_k)$$

Approximation

If roots of the denominator are rational, we achieve convergence acceleration up to $\sum_{n=1}^{\infty} O(e^{-\pi n})$.