IV International Conference "Computer Assisted Mathematics" CAM-2025

Study of coefficients of finite Dirichlet series vanishing at some zeros of Riemann's zeta function

Authors: Roman Cherepanov, Dmitry Emelyanov, Maksim Subbotin.
Idea by: Yuri Matiyasevich.

Introduction (1/2)

The idea of approximating the Riemann zeta function, using finite Dirichlet series vanishing at initial zeros of the zeta function, was described by Gleb Beliakov and Yuri Matiyasevich. Let s_k denote the initial nontrivial zeros of the zeta function, and a_i denote the coefficients of the series to be evaluated. The method proposed involves the solution of a SLAE:

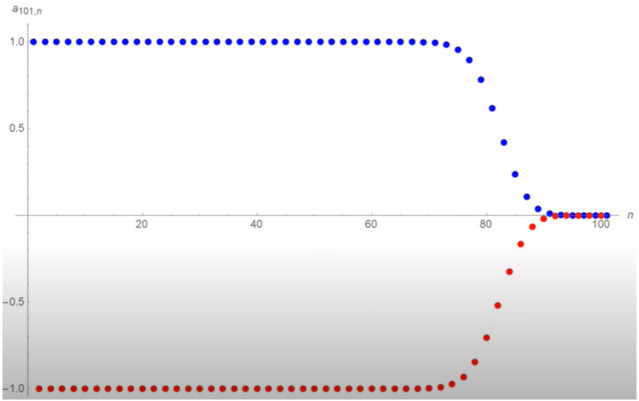
$$\begin{cases} a_1 \cdot 1^{-s_1} + a_2 \cdot 2^{-s_1} + \cdots + a_m \cdot m^{-s_1} = 0; \\ a_1 \cdot 1^{-s_2} + a_2 \cdot 2^{-s_2} + \cdots + a_m \cdot m^{-s_2} = 0; \\ \ldots \\ a_1 \cdot 1^{-s_n} + a_2 \cdot 2^{-s_n} + \cdots + a_m \cdot m^{-s_n} = 0. \end{cases}$$

In order to exclude the trivial solution with $a_i = 0$, an additional restriction is imposed on the system: $a_1 = 1$. Then, to ensure the matrix is square, $m := n + 1$.

1. The initial coefficients of the series obtained resemble the alternating zeta function.

2. Hypothesis: the result series is similar to

$$(b_1 + b_2 \cdot 2^{-s})\zeta(s)$$



Extending the method to several coefficients

For a system of equations with several fixed coefficients and the same number of zeros, the number of the coefficients increases. If n zeros are used and f coefficients are fixed, the number of rows and columns equals $N = n + f$.
In matrix form, for fixed $a_1 = 1, a_2 = 0, a_3 = 0.5$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 1^{-s_1} & 2^{-s_1} & 3^{-s_1} & 4^{-s_1} & \dots & n^{-s_1} \\ 1^{-s_2} & 2^{-s_2} & 3^{-s_2} & 4^{-s_1} & \dots & n^{-s_2} \\ \dots & & & & & \\ 1^{-s_{N-3}} & 2^{-s_{N-3}} & 3^{-s_{N-3}} & 4^{-s_{N-3}} & \dots & n^{-s_{N-3}} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \dots \\ a_N \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

Fixing first f coefficients (1/2)

- The initial coefficients tend to form horizontal lines, meaning they have approximately the same values;
- For 2 fixed coefficients, 4 lines are obtained (1);
- How many lines will be obtained from 3 fixed coefficients?

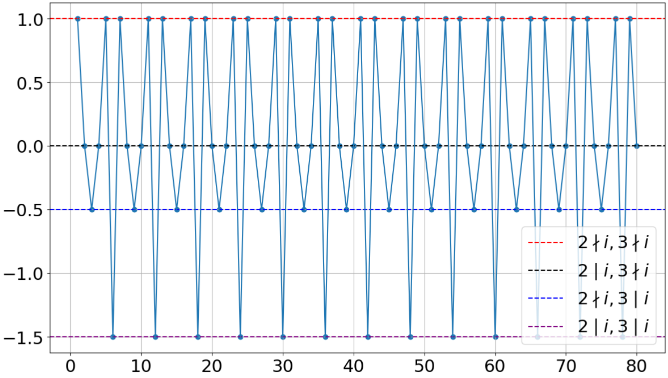


Figure 1: Fixed $a_1 = 1, a_2 = 0$

Fixing first f coefficients (2/2)

- For 3 fixed coefficients, 6 lines are obtained;
- It is discovered that each line contains coefficients that belong to the same *divisibility class*;
- For f fixed coefficients, if indices i and j share divisibility by $2, 3, \dots, f+1$, coefficients a_i and a_j are approximately equal.

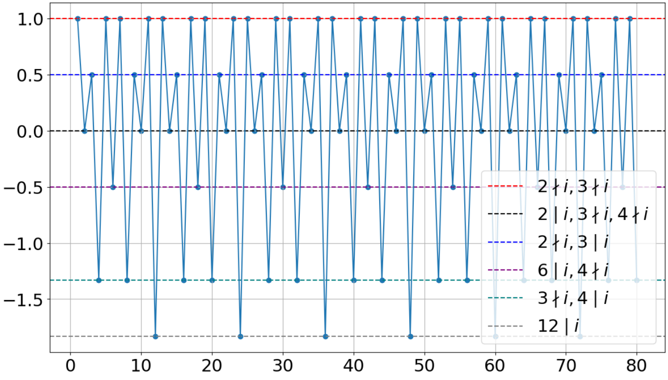


Figure 2: Fixed $a_1 = 1, a_2 = 0, a_3 = 0.5$

Divisibility classes

With increasing the number of fixed coefficients f, the maximum number of divisibility classes increases or remains the same.

1. When $f + 1$ is a prime number, each class contains elements that are divisible or not divisible by $f + 1$;
2. When $f + 1$ is a power of prime p^k, only classes that contain p^{k-1} are divided into two classes;
3. When $f + 1$ is not a power of prime, no additional divisibility classes are added. E.g. if $f + 1 = 6$, divisibility by 6 is defined by 2 and 3;
4. As a result, for f fixed coefficients c_f divisibility classes are obtained, where:

$$c_f = \begin{cases} 2, & \text{if } f = 1; \\ \frac{k+1}{k} \cdot c_{f-1}, & \text{if } f+1 \text{ — prime to the power } k; \\ c_{f-1}, & \text{if } f+1 \text{ — composite, not prime to the power.} \end{cases}$$

f	c_f
1	2
2	4
3	6
4	12
5	12
6	24
7	32
8	48
9	48

Table 1: c_f growth

Fixing coefficients from the same class

With restrictions applied to the coefficients from the same class, the solution diverges:

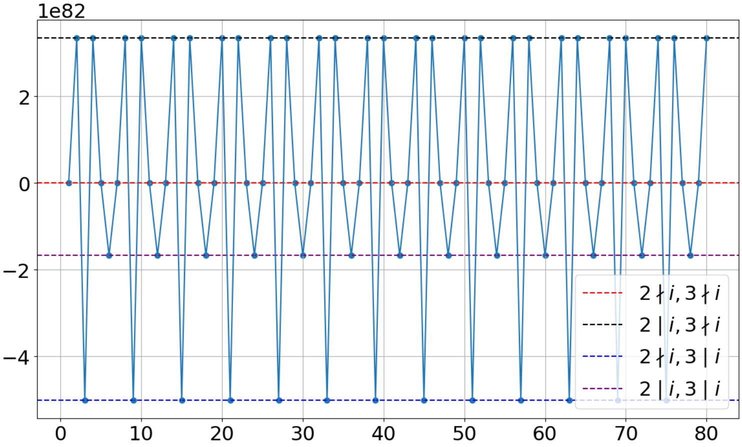


Figure 3: Fixed $a_1 = 1, a_5 = 0$

Merging classes

The initial coefficients of a series may be restricted to behave like the coefficients of a series, constructed with a less number of fixed coefficients, without divergence:

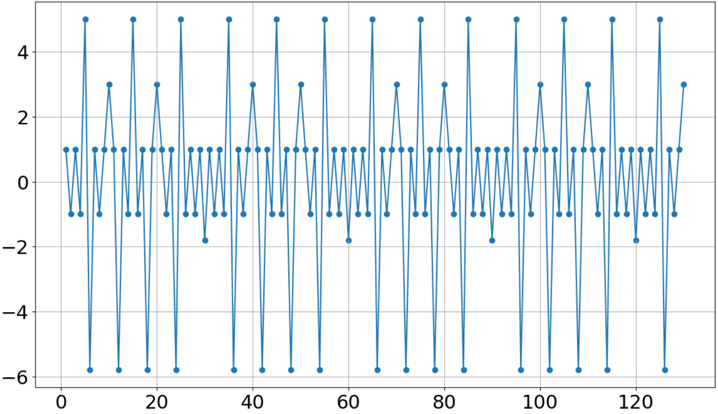


Figure 4: Fixed $a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1, a_5 = 5$

Unexpected divergence with different classes restricted

In some cases the solution diverges, even if the restrictions imposed refer to different divisibility classes:

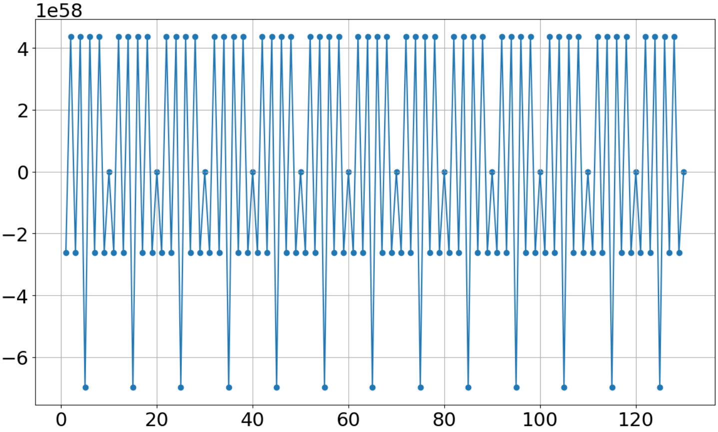


Figure 5: Fixed $a_{60} = 1, a_{10} = 2, a_{20} = 3, a_{30} = 4$

Evaluating polynomials describing series obtained (1/2)

Reminder: for 1 fixed coefficient, the result series behaves like $(b_1 + b_2 \cdot 2^{-s})\zeta(s)$.
For f fixed coefficients, the hypothesis is that the initial coefficients of the result series resemble such a product:

$$\zeta(s)\sum_{k=1}^{f+1} b_k k^{-s}.$$

For example, with fixed $a_1 = 1, a_2 = 0$ (1), the first six coefficients of the series are:

$$1^{-s} + 0 \cdot 2^{-s} + (-\frac{1}{2}) \cdot 3^{-s} + 0 \cdot 4^{-s} + 1 \cdot 5^{-s} + (-\frac{3}{2}) \cdot 6^{-s}.$$

The first six coefficients of the product $(b_1 \cdot 1^{-s} + b_2 \cdot 2^{-s} + b_3 \cdot 3^{-s})\zeta(s)$ are:

$$b_1 \cdot 1^{-s} + (b_1 + b_2) \cdot 2^{-s} + (b_1 + b_3) \cdot 3^{-s} + (b_1 + b_2) \cdot 4^{-s} + b_1 \cdot 5^{-s} + (b_1 + b_2 + b_3) \cdot 6^{-s}.$$

Evaluating polynomials describing series obtained (2/2)

The corresponding system of equations is:

$$\begin{cases} 1 = b_1, \\ 0 = b_1 + b_2, \\ -\frac{1}{2} = b_1 + b_3, \end{cases} \implies \begin{cases} b_1 = 1, \\ b_2 = -1, \\ b_3 = -\frac{3}{2}. \end{cases}$$

In a general case, a coefficient a_i is equal to the sum of coefficients b_j, where $j \mid i$. Such a system of equations leads to evaluating all values b_j. For example, for 5 fixed coefficients:

$$\begin{cases} a_1 = b_1 \\ a_2 = b_1 + b_2 \\ a_3 = b_1 + b_3 \\ a_4 = b_1 + b_2 + b_4 \\ a_5 = b_1 + b_5 \\ a_6 = b_1 + b_2 + b_3 + b_6 \end{cases} \implies \begin{cases} b_1 = a_1 \\ b_2 = a_2 - b_1 \\ b_3 = a_3 - b_1 \\ b_4 = a_4 - b_1 - b_2 \\ b_5 = a_5 - b_1 \\ b_6 = a_6 - b_1 - b_2 - b_3 \end{cases} \implies \begin{cases} b_1 = a_1 \\ b_2 = a_2 - a_1 \\ b_3 = a_3 - a_1 \\ b_4 = a_4 - a_2 \\ b_5 = a_5 - a_1 \\ b_6 = a_6 - a_2 - a_3 + a_1 \end{cases}$$

Note on unexpected divergence

With the analysis of system of equations for b_k conducted, the unexpected divergence (5) may be explained. With 4 fixed coefficients, there are 5 coefficients b_k. The system of equations is:

$$\begin{cases} a_{10} = b_1 + b_2 + b_5; \\ a_{20} = b_1 + b_2 + b_4 + b_5; \\ a_{30} = b_1 + b_2 + b_3 + b_5; \\ a_{60} = b_1 + b_2 + b_3 + b_4 + b_5. \end{cases} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

One can easily observe that the rank of the matrix is 3. 4 restrictions can not be imposed arbitrarily on such a system.
Note that fixing coefficients from the same class also decreases the rank of the matrix, which leads to inconsistency if the augmented's matrix rank is not decreased accordingly.

Conclusion

The study discovered several indices configurations affecting series constructed using the initial zeros of the zeta function.

- When f coefficients are fixed, the initial coefficients share values if they share divisibility by $2, 3, \dots, f+1$;
- The series obtained resembles a product $\zeta(s)\sum_{k=1}^{f+1} b_k k^{-s}$;
- If the system of equations for a_k is not consistent, the result series diverges;
- As a partial case, imposing restrictions on coefficients from the same divisibility class leads to divergence.