Study of Coefficients of Finite Dirichlet Series Vanishing at Some Zeros of Riemann's Zeta Function

Roman Cherepanov, Dmitry Emelyanov and Maksim Subbotin

Abstract. In 2013, Yu. V. Matiyasevich numerically studied finite Dirichlet series vanishing at multiple non-trivial zeros of the Riemann zeta function. He fixed the first coefficient of these series to 1 and observed that the initial coefficients of the considered Dirichlet series closely resemble those of the alternating zeta function.

This study was extended by fixing multiple coefficients of such finite Dirichlet series, and patterns were identified in the remaining initial coefficients. Specifically, these coefficients approximate the coefficients of the product: $\zeta(s)\sum_{k=1}^{f+1} b_k k^{-s}$, where f denotes the number of fixed coefficients, and the b_k values are determined from the series coefficients. The results of this analysis provide new insights into the relationship between the Riemann zeta function and number theory.

Introduction

The idea of approximating the Riemann zeta function, $\zeta(s)$, using finite Dirichlet series that vanish at initial zeros of the zeta function, was described in [1].

This involves constructing a Dirichlet series $R(s) = \sum_{k=1}^{N} a_k k^{-s}$, where N denotes the number of coefficients, such that $R(s)$ vanishes at the first n non-trivial zeros of the zeta function. To ensure a non-trivial solution, one coefficient should be fixed (e.g. $a_j = 1$), which is enforced via an additional equation.

This system can be represented in matrix form as $\mathbf{Sa} = \mathbf{b}$, where $\mathbf{S}$ is an $N \times N$ matrix, $\mathbf{a}$ is a column vector of size N, containing the unknown coefficients, and $\mathbf{b}$ is a column vector of size N with constant terms.

The number of coefficients N required for the series construction depends on the number of constraints the series should satisfy. To ensure a square matrix, the number of constraints must equal N. For example, n constraints arise from vanishing at n points, and one constraint is imposed to fix the first coefficient, yielding $N = n + 1$. This method was detailed by Yu. V. Matiyasevich in [1, 3].

The present study focuses on exploring series with **several** fixed coefficients. Let f denote the number of fixed coefficients. Then $N = n + f$, as n constraints are imposed for vanishing at n points and f constraints are introduced for fixing f coefficients. For instance, for a series with three fixed coefficients $a_1 = 1, a_2 = 0, a_3 = 0.5$, the system in (1) is solved.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 1^{-s_1} & 2^{-s_1} & 3^{-s_1} & 4^{-s_1} & \dots & n^{-s_1} \\ 1^{-s_2} & 2^{-s_2} & 3^{-s_2} & 4^{-s_1} & \dots & n^{-s_2} \\ \dots & & & & & \\ 1^{-s_{n-f}} & 2^{-s_{n-f}} & 3^{-s_{n-f}} & 4^{-s_{n-f}} & \dots & n^{-s_{n-f}} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ \dots \\ 0 \end{bmatrix} \tag{1}$$

Using this procedure, a finite Dirichlet series with any number of fixed coefficients can be constructed. In this study, the non-trivial zeros of the Riemann zeta function, computed by Yu. V. Matiyasevich and described in [2], are employed.

Main Results

The coefficient values of specific series are analyzed here. All series in this paper are constructed using the first 400 zeta zeros with a precision of 600 decimal digits. The focus is on the initial coefficients of the series due to the increasing magnitude observed toward the middle and beyond.

The first 80 coefficients of the series with 2 and 3 initial coefficients fixed are displayed in Figures 1 and 2.

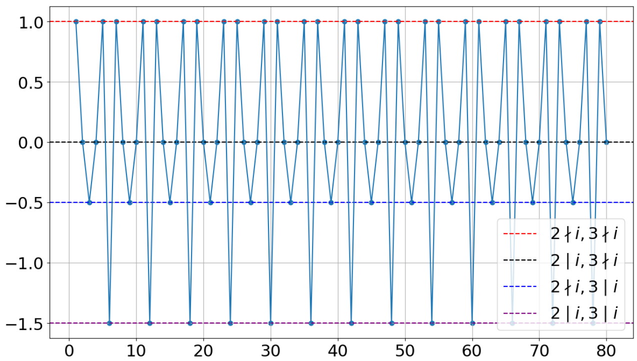

FIGURE 1. Series initial coefficients for fixed $a_1 = 1, a_2 = 0$

Observations indicate that the coefficient values form distinct groups. For instance, in Figure 1, four distinct classes of coefficient magnitudes are evident: their values are close to $1, 0, -0.5, -1.5$.

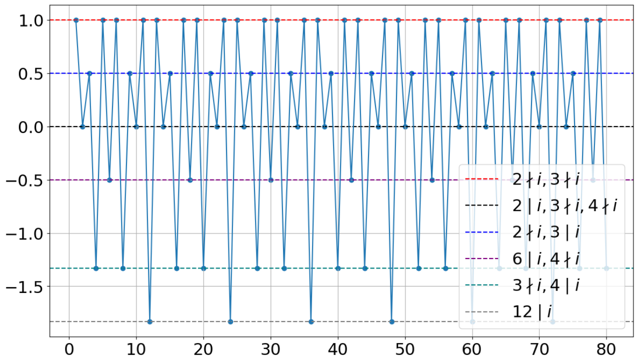

FIGURE 2. Series initial coefficients, fixed $a_1 = 1, a_2 = 0, a_3 = 0.5$

Similarly, when three initial coefficients are fixed, six classes of coefficients emerge. The pattern emerging suggests that the number of coefficient classes, when f initial coefficients are fixed, is equal to the number of possible divisibility combinations by integers $2, 3, \ldots, f + 1$.

For other sets of fixed coefficients' indices, behavior of the constructed series may vary. If indices from the same divisibility class are constrained to have different values, the result series values' magnitudes are larger than expected, as shown in Figure 3.

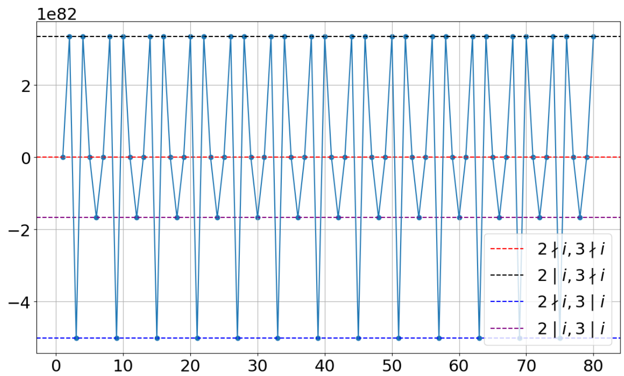

FIGURE 3. Initial coefficients of the series with $a_1 = 1, a_5 = 0$

Another possible variation in the result series behavior is a smaller number of coefficients' value magnitudes. Under certain constraints, the result series appears to resemble a series, constructed from a smaller number of constraints, as indicated in Figure 4, where three fixed coefficients lead to four coefficient magnitude classes.

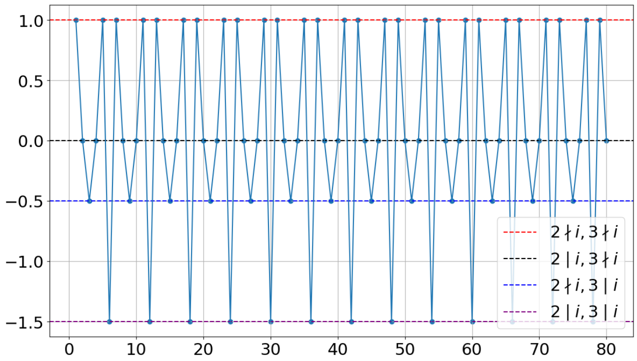

FIGURE 4. Series initial coefficients for fixed $a_1 = a_5 = 1, a_2 = 0$

After the constructed series investigation, the coefficients of the first series with $a_1 = 1, a_2 = 0$ are compared to the coefficients of the finite part of the Riemann zeta function. Under the hypothesis that the initial coefficients of the series $R(s)$ approximate the product:

$$(b_1 \cdot 1^{-s} + b_2 \cdot 2^{-s} + b_3 \cdot 3^{-s})\zeta(s),$$

the constants b_1, b_2, b_3 can be determined by analyzing the first six coefficients of the constructed series.

The first six coefficients of the series are:

$$1^{-s} + 0 \cdot 2^{-s} + (-\frac{1}{2}) \cdot 3^{-s} + 0 \cdot 4^{-s} + 1 \cdot 5^{-s} + (-\frac{3}{2}) \cdot 6^{-s}.$$

The first six coefficients of the product $(b_1 \cdot 1^{-s} + b_2 \cdot 2^{-s} + b_3 \cdot 3^{-s})\zeta(s)$ are:

$$b_1 \cdot 1^{-s} + (b_1 + b_2) \cdot 2^{-s} + (b_1 + b_3) \cdot 3^{-s} + (b_1 + b_2) \cdot 4^{-s} + b_1 \cdot 5^{-s} + (b_1 + b_2 + b_3) \cdot 6^{-s}.$$

The corresponding system of equations is:

$$\begin{cases} 1 = b_1, \\ 0 = b_1 + b_2, \\ -\frac{1}{2} = b_1 + b_3, \end{cases} \implies \begin{cases} b_1 = 1, \\ b_2 = -1, \\ b_3 = -\frac{3}{2}. \end{cases} \tag{2}$$

Thus, the series $R(s)$ exhibits similarity to $(1 - 2^{-s} - \frac{3}{2} \cdot 3^{-s})\zeta(s)$ in its initial coefficients.

In a general case, a coefficient of the result series a_i is equal to the sum of coefficients b_j, where $j \mid i$. Such a system of equations leads to evaluating all values b_j. For example, for five fixed coefficients the system of equations for the six b_k coefficients is:

$$\begin{cases} a_1 = b_1 \\ a_2 = b_1 + b_2 \\ a_3 = b_1 + b_3 \\ a_4 = b_1 + b_2 + b_4 \\ a_5 = b_1 + b_5 \\ a_6 = b_1 + b_2 + b_3 + b_6 \end{cases} \implies \begin{cases} b_1 = a_1 \\ b_2 = a_2 - b_1 \\ b_3 = a_3 - b_1 \\ b_4 = a_4 - b_1 - b_2 \\ b_5 = a_5 - b_1 \\ b_6 = a_6 - b_1 - b_2 - b_3 \end{cases} \implies \begin{cases} b_1 = a_1 \\ b_2 = a_2 - a_1 \\ b_3 = a_3 - a_1 \\ b_4 = a_4 - a_2 \\ b_5 = a_5 - a_1 \\ b_6 = a_6 - a_2 - a_3 + a_1 \end{cases}$$

Conclusion

This study identifies fixed index configurations that affect the series constructed using the initial zeros of the zeta function. A range of series with varying f fixed coefficients combinations was analyzed, leading to the following conclusions:

- Coefficient magnitudes depend on the divisibility of their indices by $2, ...f+1$.
- Coefficients sharing divisibility patterns exhibit nearly identical values.
- Initial coefficient values align closely with the series $R(s) \sim \zeta(s) \sum_{k=1}^{f+1} b_k k^{-s}$, where f denotes the number of fixed coefficients, and the b_k values are derived from the series coefficients.

References

[1] Gleb Beliakov & Yuri Matiyasevich, *Approximation of Riemann's Zeta Function by Finite Dirichlet Series: A Multiprecision Numerical Approach*, Experimental Mathematics **24:2** (2015), 150-161.

[2] Yu. Matiyasevich, *Calculation of Riemann's zeta function via interpolating determinants.* Preprint of Max Planck Institute for Mathematics in Bonn **18** (2013), 31.

[3] Yu. Matiyasevich, *Riemann's zeta function and finite Dirichlet series*, St. Petersburg Math. J. **27:6** (2016), 985–1002.

Roman Cherepanov
Saint Petersburg Electrotechnical University "LETI"
Saint Petersburg, Russia
e-mail: `romacherepanov2002@gmail.com`

Dmitry Emelyanov
Information Technologies, Mechanics and Optics University
Saint Petersburg, Russia
e-mail: `dima31120251@gmail.com`

Maksim Subbotin
Saint Petersburg Electrotechnical University "LETI"
Saint Petersburg, Russia
e-mail: `maksim@entroforce.ru`