Inference in Logic Theories

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Mathematical Logic

Logic: language, models, calculus

- Language defines the syntax of a logic via a formal grammar
- Model theory defines the semantics of a logic
- Calculus specifies what formulas belong to a logic by means of axioms and inference rules.

Examples:

Classical logic, intuitionistic logic, modal and temporal logics
Other nonstandard logics: argumentation, quantum, fuzzy, etc

Logic Theory: logic plus nonlogical axioms

Logic theories formalize AI systems

Mathematics is an ever-expanding deep hierarchies of logic theories

Language of first-order logics

Atoms, terms, substitution of terms for object variables

Conjunction, disjunction, implication, negation, and other connectives

Universal and extensional quantifiers over object variables

No references to concrete predicates, functions, or domains

Models

- Truth values are a subset $\mathcal{T}$ of interval $[0, 1]$
- Predicate/function interpretations define truth values of predicates and functions with constant arguments:

  $p(x_1 ... x_k) : \mathcal{D}_{x_1} \times ... \times \mathcal{D}_{x_k} \to \mathcal{T}$ $\qquad$ $f(x_1 ... x_n) : \mathcal{D}_{x_1} \times ... \times \mathcal{D}_{x_n} \to \mathcal{D}_f$
- Truth functions for logical connectives
- Specification of truth values of quantifier formulas and formulas with free variables

Calculi

- Axioms define certain formulas of a logic

  Example: $A \supset A$

- Inference rules specify how to produce new formulas of a logic from other formulas of this logic

  Example: $$\frac{A \quad A \supset B}{B}\ MP$$

- Logical axioms and inference rules define properties of logical connectives and quantifiers

Theories

Nonlogical axioms refer to concrete predicates, functions, and domains

Mathematical theories

- Equality

  $x = x \qquad (x = y \wedge A(x)) \supset A(y)$

- Mathematical induction for well-founded sets

  $(A(0) \wedge \forall x(A(x) \supset A(x+1))) \supset A(x)$

- Arithmetic (addition, multiplication)

Logic Theories

We consider logic theories of first-order logics whose nonlogical axioms do not contain metavariables, and thus all symbols in nonlogical axioms are concrete (interpreted) functions and predicates along with object variables

- Nonlogical axioms express domain knowledge, they specify properties of concrete functions and predicates for a set of domains
  Example: $t(x, y) \wedge t(y, z) \supset t(x, z)$
- Nonlogical axioms define predicates (relations, notions) via other predicates in relevant domains or inductively define them (not real inductive definitions!)
  Examples: $p(x) \equiv \neg(y = 1) \wedge \neg(z = 1) \wedge y \cdot z = x$
  $e(x) \equiv x = 0 \vee (x \geq 2 \wedge e(x - 2))$
- Other functions and predicates are implemented by computer programs, their formulas with constant arguments are dynamically generated nonlogical axioms
  Example: $fact(3) = 6$

Logical Formalisms and Inference Methods

Calculus formalisms

- Hilbert-style (axioms)
- Gentzen-style (introduction inference rules)
- Natural deduction (introduction and elimination inference rules)
- etc

Efficient inference methods

- Resolution method (for classical first-order logic and formulas in clausal form)
- Tableau methods
- Inverse method
- Focusing methods

Inference in Logics and Theories

- Inference in nonstandard logics
- Inference in mathematical theories
- Inference with using tons of nonlogical axioms (hierarchically defined)
- Cut admissibility, subformula property, analytic calculi, normal forms
- Focusing methods have been extended onto logical theories, resolution methods are also applicable to logic theories
- Axioms vs inference rules, geometric axioms, bipolar inference rules

$$P_1 \wedge \ldots \wedge P_k \supset Q_1 \vee \ldots \vee Q_m$$

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Artifial Intelligence

- Past: symbolic, implemented by inference in logic theories
- Present: numerical, implemented by neural networks

Neural network AI

$\mathcal{V}$ - the set of real-valued vectors

$$\mathcal{D} \xrightarrow{\text{embedding}} \mathcal{V} \xrightarrow{\text{neural network}} \mathcal{V} \xrightarrow{\text{translation}} \mathcal{D}'$$

Modern AI should be explainable!

Explanations are deductions in a logic theory

Neural networks are not partial recursive functions

Fuzzy Knowledge Systems

- Fuzzy logics rely on truth functions whose domain and range is interval $[0, 1]$

- Fuzzy knowledge systems can be specified as logic theories

- Computation in fuzzy knowledge systems can be implemented as inference in logic theories

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Dombi, J., Jonas, T. Approximate reasoning based on the preference implication. Fuzzy Sets and Systems, v. 499, 2025