Guided Discovery with Dynamic Mathematics

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Abstract. One of the most important advantages of interactive mathematical systems as a teaching tool is their ability to support mathematical experiments. Such experiments make it possible to present many topics of school mathematics in the "guided discovery" format. We give several examples of activities showcasing various types of discovery scenarios and their implementation based on MathKit environment.

Introduction

The concept of "learning through discovery" [1], introduced by J. Bruner in the 1960s and later evolving into "guided discovery," received a powerful impetus in mathematics teaching with the advent of interactive mathematical systems (IMS) - originally dynamic geometry software. IMS provided significantly broader opportunities to teach and learn by conducting mathematical experiments, through which students acquire new knowledge independently rather than receiving it in ready-made form.

When organizing such research, the teacher builds so-called "scaffolding" - a system of (guiding) hints and leading questions that guide the student along the desired path to the desired result. Obviously, the more detailed and straightforward the hints, the less creative component in the student's work with the task, and therefore the smaller the emotional effect of the "discovery," and vice versa. On the other hand, the level of detail of the hint system should correspond to the level of the students for whom they are intended.

We present examples of guided discovery scenarios of different types, showing how one can "discover" not only visual, observable geometric facts but also quite complex formulas. Models based on these scenarios can be found in the collections of educational materials created in the MathKit environment.

1. Examples of Guided Discovery

In the first two examples, which can be called classical - as they have been implemented many times in various IMS - the "discovery" occurs in the simplest and most natural way - through the search for invariants under variation, i.e., by observing the behavior of a geometric figure when dragging its initial points to notice those properties that remain unchanged.

1.1. Example 1: Theorem on Triangle Medians

The model - a triangle with medians - arises naturally when getting acquainted with the definition of a median. The invariant property (the fact that medians always intersect at one point) is striking. Much less obvious is the second statement of the theorem - that medians are divided by their intersection point in a constant ratio of 2:1. Usually, students are led to discover this fact straightforwardly - by measuring the lengths of median segments (a typical example is [2]). But there is a more subtle hint - to mark the midpoints of the segments from the centroid to the vertices: during variation, it is clearly noticeable that after this, each median is divided into three equal parts.

1.2. Example 2: Napoleon's and van Aubel's Theorems

The model for "discovering" Napoleon's theorem is a triangle on whose sides "external" equilateral triangles and their centers are constructed. What can be said about the arrangement of the centers? It is easier to guess the answer if you connect these centers with segments and move the original triangle: usually everyone notices that the triangle of centers is also always equilateral (this is the theorem).

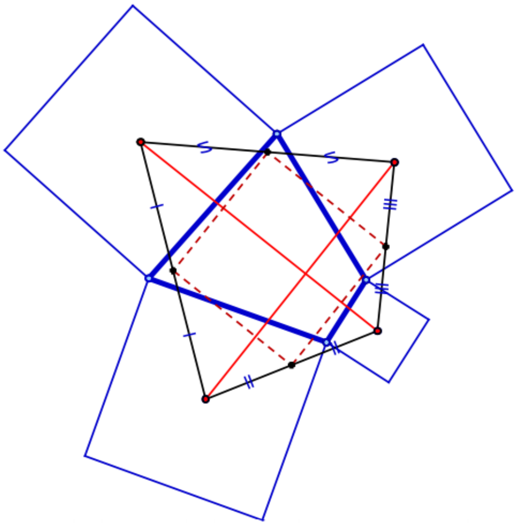

FIGURE 1. On van Aubel's Theorem

Identifying the properties of a similar construction - a quadrilateral with vertices at the centers of squares constructed on the sides of an arbitrary quadrilateral - turns out to be significantly more difficult, even if given another hint - to draw its diagonals. They will be perpendicular and equal (this is van Aubel's theorem), but if students still see perpendicularity, they very rarely see equality. A good, although perhaps too obvious hint is to construct Varignon's parallelogram for the quadrilateral with vertices at the centers of squares - it is a square, and this is equivalent to van Aubel's theorem. The van Aubel configuration possesses many remarkable properties. Finding (and proving) as many of them as possible is a topic for meaningful creative work that goes beyond guided discovery.

2. Discovering Formulas

Let's move on to examples where the invariant is not a geometric property, such as concurrency of lines or the shape of a triangle, but a relationship between quantities. How to organize the discovery of a formula?

2.1. Example 3: Sum of Angles of a Triangle

Numerous educational models dedicated to the theorem on the sum of angles of a triangle are posted on Internet resources. As a rule, this result is presented in ready-made form: the angles of the triangle are measured and their sum is calculated, and the student checks that it does not depend on the triangle by changing it.

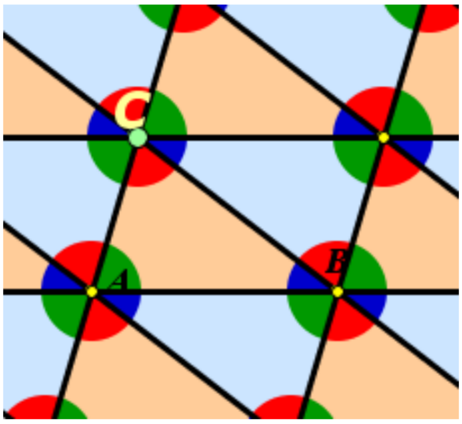

FIGURE 2. On the Sum of Angles of a Triangle

Occasionally, this scenario is clothed in a visual geometric form [3]: three corners are assembled into a straight angle. In more interesting versions, students are asked to measure and add the angles independently. In any case, the task of adding three angles, algebraically or geometrically, comes from the teacher; the theorem is, as it were, imposed on the students.

In our version of guided discovery of this theorem, initially nothing is said about angles. A triangle is given and buttons that generate its copies, obtained either by shifting or by rotating by 180°. With these copies, one must tile the plane (part of it). Then it is checked that any triangle can be taken as a tiling tile, and the question is posed: what property of the triangle ensures the possibility of tiling without gaps and overlaps? (The doubled sum of the angles of the triangle equals 360°.) The discovery of the theorem turns out to be a byproduct of another, but quite meaningful activity.

2.2. Example 4: Menelaus' Theorem

This theorem gives a condition for the collinearity of three points taken on the lines bounding a triangle, expressed by a rather complex formula. How to lead students to its "discovery" without hinting at what it looks like? Here is a possible scenario [4].

1. Everything starts with a very simple construction - a triangle ABC and a line intersecting its sides AB and BC at points M and K. The goal of the research is not initially communicated to the students. They simply need to formulate a hypothesis about the behavior of the line when moving vertex B (namely, that it always passes through the same point L on line AB). Hint: enable drawing the trace of the line.
2. Conclusion: the position of point L on AC is determined by the positions of points M and K on AB and BC. The positions of points are numerically given by simple ratios of triples of points $m = AM/MB$, $k = BK/KC$, $l = CL/LA$. Any two of them determine the third. The next task is to understand what this dependence looks like.
3. If you move M along line AB without touching K, then L will move along AC. Let's construct a point $(m; l)$ on the coordinate plane and the curve that it describes when M moves along line AB, i.e., essentially, the graph of the dependence of l on m. We get a hyperbola, or rather, a curve resembling a hyperbola.
4. Let's check if it is really a hyperbola by computing ml and making sure that this product is constant (but depends on the position of point K). So, $ml = c(k)$.
5. It remains to find the function $c(k)$. You can again construct its graph, but this will not be necessary if you notice that the product of all three ratios $kml = kc(k)$ depends only on k. And since by construction points K and M are equivalent, it can only depend on m. This means that it does not depend on the arrangement of points at all: $kml = const$. Multiplying the measured ratios, we get: $kml = -1$ (MathKit computes ratios taking into account the sign). This is Menelaus' theorem.

According to a similar scenario, one can organize guided discovery of many theorems, including the Pythagorean theorem: fix one leg of the triangle and construct a graph of the dependence of the area S of the square on the hypotenuse on the area s of the square on the second leg. The result will be a straight line with a slope of 1, whence $S = s + const$, and it remains only to understand that the constant here is the area of the square on the first leg. It is even easier to detect the form of the dependence if instead of its graph one uses the so-called dipograph: two parallel number axes and a segment connecting point s of one axis to point S of the other.

Conclusion

The search for new scenarios of guided discovery in different areas of school mathematics, improving the tools necessary for this, is one of the important directions of development of the MathKit environment and the educational models created with its help.

References

[1] Bruner, J. S., *The act of discovery.* Harvard Educational Review, 31(1), 21–32, 1961.

[2] kathleenh, *Median Theorem*, available at `https://www.geogebra.org/m/cugdgg3p`.

[3] Geogebra Content Team, *Sum of Interior Angles of a Triangle*, available at `https://www.geogebra.org/m/u23bahu7`.

[4] *How to Make a Mathematical Discovery*, available at `https://obr.1.c.ru/mathkit/lessons5.html`.

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